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A decomposition of the anisotropic harmonic oscillator with rationally related frequencies

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Abstract. The degeneracies in the energy levels of the n dimensional anisotropic harmonic oscillator are examined and related to the representations of SU(n). The space on which the oscillator acts is then decomposed so that the hamiltonian splits up into the direct sum of isotropic oscillators. This decomposition fits with the pattern of degeneracies noted earlier.

1. Introduction

If $H(\boldsymbol{\omega}) = H(\omega_1, \ldots, \omega_n)$ represents the hamiltonian of the n dimensional harmonic oscillator with frequencies ω_j, the solutions of

$$H(\boldsymbol{\omega})\psi = E^{\boldsymbol{\omega}}\psi \tag{1}$$

are given by

$$E^{\boldsymbol{\omega}} = E^{\boldsymbol{\omega}}(\boldsymbol{m}) = (m_1\omega_1 + \ldots + m_n\omega_n), \tag{2}$$

where each m_j can be any positive integer.

In the case of the isotropic oscillator each $\omega_j = M$ and so

$$E = E(\boldsymbol{m}) = M\left(\sum_{j=1}^{n} m_j\right) \tag{3}$$

which is clearly degenerate for all dimensions greater than 1. Indeed the degeneracy spaces of the n dimensional isotropic oscillator are in 1–1 correspondence with the irreducible totally symmetric representations of SU(n), and various reasons can be given for the connection between SU(n) and the isotropic oscillator (Baker 1956, Jauch and Hill 1940).

The object of this work is to discuss the degeneracy in the case when the frequencies are rationally related. In § 2 it is shown how the degeneracies correspond to representations of SU(n) and in § 3 the anisotropic oscillator is decomposed in a way which explains this correspondence.

2. The anisotropic oscillator

The reasons usually given for the connection between SU(n) and the isotropic harmonic oscillator break down when the ω_j are allowed to vary. For instance, the infinitesimal generators of SU(n) can still be represented by combinations of the creation and annihilation operators, as in Jauch and Hill (1940) but these generators only commute with the oscillator hamiltonian when the frequencies are equal. Or again, the oscillator hamiltonian can still be written in a complex form as in Baker (1956) and the unitary transformations are still canonical but when the ω_j vary, these transformations do not always leave the oscillator hamiltonian invariant. However, several authors, including Demkov (1963) and Vendramin (1968), have noted that there is a connection between unitary groups and anisotropic harmonic oscillators and so let us consider the degeneracy spaces of such systems.

Degeneracies of the energy level occur when there are nontrivial solutions of

$$\sum_{j=1}^{n} \omega_j m_j = 0, \tag{4}$$

which implies a rational relation between the frequencies. Clearly there is greatest scope for degeneracy when the frequencies are rationally related, that is, when each ω_j/ω_n is a rational number. Complications arise if some of the ω_j are rational combinations of the others, but such cases are equivalent to the sum of various linked rationally related systems of lower dimension.

Thus, the only systems of present interest are those with rationally related frequencies, and then it is no loss in generality to multiply all the frequencies by a constant so that they are integers with HCF equal to 1. Assume that the LCM of these frequencies is M, and define

$$d_j = M\omega_j^{-1}; \qquad \boldsymbol{d} = (d_1, \ldots, d_n) \qquad \text{and} \qquad d = \prod_{j=1}^{n} d_j.$$

Using the euclidean algorithm on each coordinate of $\boldsymbol{m}$ we can write

$$m_j = \tilde{m}_j d_j + r_j$$

where $r_j, \tilde{m}_j \in Z$ and $m_j, \tilde{m}_j \geqslant 0$, $0 \leqslant r_j \leqslant (d_j - 1)$. From now on this will be written as

$$\boldsymbol{m} = \tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r} \tag{5}$$

where $\tilde{\boldsymbol{m}} = (\tilde{m}_1, \ldots, \tilde{m}_n)$ and $\boldsymbol{r} = (r_1, \ldots, r_n)$.

The energy levels are given by $E(\boldsymbol{m}) = (\boldsymbol{\omega}, \boldsymbol{m})$ so that

$$\begin{aligned} E(\boldsymbol{m}) &= E(\tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r}) = (\boldsymbol{\omega}, (\tilde{\boldsymbol{m}} \,.\, \boldsymbol{d})) + (\boldsymbol{\omega}, \boldsymbol{r}) \\ &= (\tilde{\boldsymbol{m}}, \boldsymbol{\omega} \,.\, \boldsymbol{d}) + (\boldsymbol{\omega}, \boldsymbol{r}) \\ &= M\left(\sum_{j=1}^{n} \tilde{m}_j\right) + (\boldsymbol{\omega}, \boldsymbol{r}). \end{aligned} \tag{6}$$

This expression is very similar to the formula for the isotropic case with frequency M; the only difference is in the extra term $(\boldsymbol{\omega}, \boldsymbol{r})$. Clearly the degeneracy of the rationally related case can be calculated from this expression.

Assume, to start with, that $(\boldsymbol{\omega}, \boldsymbol{r}) \not\equiv (\boldsymbol{\omega}, \boldsymbol{r}') \bmod M$ unless $\boldsymbol{r} = \boldsymbol{r}'$ and note that from the definition there are d different $\boldsymbol{r}$ allowed. Thus for each of the d values of $\boldsymbol{r}$, there is a series of energy levels as $\tilde{\boldsymbol{m}}$ varies and the term $(\boldsymbol{\omega}, \boldsymbol{r})$ ensures that for different values of $\boldsymbol{r}$ these series of levels are completely disjoint. Now take any $\boldsymbol{r}$ and hold it fixed, then by (6) the degeneracy of $E(\tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r})$ is the same as the degeneracy of the level $E(\tilde{\boldsymbol{m}})$ in an isotropic oscillator. But the degeneracy of $E(\tilde{\boldsymbol{m}})$ corresponds to an irreducible representation of SU(n) and so the degeneracies in each of the d series are determined by the irreducibles of SU(n).

In some cases (when $n > 2$) it is possible that $(\boldsymbol{\omega}, \boldsymbol{r}) \equiv (\boldsymbol{\omega}, \boldsymbol{r}') \bmod M$ even when $\boldsymbol{r}$ and $\boldsymbol{r}'$ are not equal, and then at least two of the d series will coincide, except perhaps near the ground state. To be precise, assume there are two such $\boldsymbol{r}$ and $\boldsymbol{r}'$, then for almost all $\tilde{\boldsymbol{m}} \in (Z^+)^n$ there is an $\tilde{\boldsymbol{m}}'$ such that $E(\tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r}) = E(\tilde{\boldsymbol{m}}' \,.\, \boldsymbol{d} + \boldsymbol{r}')$. This means that this energy E will occur in at least two of the series and so the degeneracy of E will be the direct sum of the degeneracy of E within the series determined by $\boldsymbol{r}$, with that within the series determined by $\boldsymbol{r}'$. Thus there may be extra 'accidental' degeneracy, corresponding to the direct sum of two or more irreducible representations of SU(n).

Theorem 1. If the energies of the rationally related harmonic oscillator $H(\boldsymbol{\omega})$ are denoted by $E^{\boldsymbol{\omega}}$ and if E is the energy in the isotropic case, then

$$E^{\boldsymbol{\omega}}(\tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r}) = ME(\tilde{\boldsymbol{m}}) + (\boldsymbol{\omega}, \boldsymbol{r}).$$

Further, the degeneracy spaces of $H(\boldsymbol{\omega})$ correspond to the representations of SU(n) similarly to the isotropic case, except that either several copies of one irreducible representation or a reducible representation may occur, and this depends exactly on the degeneracy with respect to $\boldsymbol{r}$ of $(\boldsymbol{\omega}, \boldsymbol{r}) \bmod M$.

3. A decomposition of the anisotropic oscillator

There are several explanations of this degeneracy in relatively simple cases, for example, Demkov (1963), Vendramin (1968), Maiella and Vilasi (1969). However, most of the relevant facts can be deduced in the general n dimensional case by considering the Hilbert space V on which the hamiltonian acts. Keeping $\boldsymbol{r}$ fixed, let

$$V^{\boldsymbol{r}} = \bigoplus_{\tilde{\boldsymbol{m}}} F_{\boldsymbol{m}};$$

where $\boldsymbol{m} = \tilde{\boldsymbol{m}} \,.\, \boldsymbol{d} + \boldsymbol{r}$ and $F_{\boldsymbol{m}}$ is the degeneracy space of $H(\boldsymbol{\omega})$ corresponding to $E^{\boldsymbol{\omega}}(\boldsymbol{m})$. There will be d distinct $V^{\boldsymbol{r}}$ formed as each r_j varies from zero to $(d_j - 1)$ and

$$V = \bigoplus_{\boldsymbol{r}} V^{\boldsymbol{r}}; \tag{7}$$

where the direct sum contains each of the distinct $V^{\boldsymbol{r}}$ exactly once.

It is possible to write down the projection operators which perform the decomposition (7). Indeed, if

$$P_{\boldsymbol{r}} = \prod_{j=1}^{n} \left\{ \prod_{r_j' \neq r_j} \left(\frac{\sin^2(a_j^* a_j - r_j')\pi d_j^{-1}}{\sin^2(r_j - r_j')\pi d_j^{-1}} \right) \right\} I,$$

where a_j^* and a_j are the creation and annihilation step operators in the jth dimension, it is routine to check that

$$P_{\boldsymbol{r}} P_{\boldsymbol{r}'} = \delta(\boldsymbol{r} - \boldsymbol{r}') P_{\boldsymbol{r}}; \qquad I = \bigoplus_{\boldsymbol{r}} P_{\boldsymbol{r}}; \qquad P_{\boldsymbol{r}} : V \to V^{\boldsymbol{r}}.$$

Observe that $V^{\boldsymbol{r}}$ is isomorphic to V as the map $T_{\boldsymbol{r}}$ from $V^{\boldsymbol{r}}$ to V defined on the basis, consisting of the oscillator eigenfunctions $f_{\boldsymbol{m}}$, by

$$T_{\boldsymbol{r}} : f_{\tilde{\boldsymbol{m}}.\boldsymbol{d}+\boldsymbol{r}} \to f_{\tilde{\boldsymbol{m}}}$$

and extended to all of $V^{\boldsymbol{r}}$ by linearity, is unitary because it takes a complete orthonormal basis in $V^{\boldsymbol{r}}$ on to a complete orthornormal basis in V.

Let $U = \oplus_r T_r P_r$, then U is unitary and $U: V \to \oplus_d V$ and so consider $UH(\omega)U^{-1}$. Let $H(I)$ represent the hamiltonian of the isotropic oscillator with unit frequency so that $H(I)f_m = (\Sigma_j m_j)f_m$, whereas $H(\omega)f_m = (\Sigma \omega_j m_j)f_m$. A direct calculation on this basis shows that the action of $UH(\omega)U^{-1}$ on $\oplus_d V$ is equivalent to the action of $\oplus_r\{MH(I)+(\omega, r)\}$; that is,

$$UH(\omega)U^{-1} = \bigoplus_r \{MH(I)+(\omega, r)\}. \tag{8}$$

Theorem 2. The anisotropic harmonic oscillator with rationally related frequencies is unitarily equivalent to the direct sum of d isotropic oscillators with frequency M and varying ground state energies (ω, r).

As the dynamical symmetry group of each $MH(I)$ is SU(n), theorem 2 provides some justification for calling SU(n) the dynamical symmetry group of $H(\omega)$, but it is not very compelling. However, theorem 2 and equation (8) demonstrate the connection with SU(n) stated in theorem 1 because they show that every symmetric representation of SU(n) will correspond to d degeneracy spaces, one arising from each operator $MH(I)+(\omega, r)$.

In fact, each of the degeneracy spaces of $MH(I)+(\omega, r)$ corresponds to a particular irreducible of SU(n) and so, if it is impossible for $(\omega, r) \equiv (\omega, r') \bmod M$, (8) shows that each degeneracy space of $H(\omega)$ corresponds to an irreducible symmetric representation of SU(n). But, if the ground states are degenerate, that is, if $(\omega, r) \equiv (\omega, r') \bmod M$, there will be extra 'accidental' degeneracy between the different $MH(I)+(\omega, r)$. Hence it may be necessary to associate a reducible representation of SU(n) with some of the degeneracy spaces of $H(\omega)$. These remarks explain the relation between the degeneracy of the rationally related harmonic oscillator and SU(n) that was stated in theorem 1.

Theorem 2 can also be related to the work of Maiella and Vilasi (1969). They consider the three-dimensional case by introducing step operators on various subspaces, which combine to yield the Lie algebra of SU(3) and which commute with the hamiltonian.

Now let

$$A_j^r = U^{-1}\{\bigoplus_{r'} a_j \delta(r-r')\}U,$$

then A_j^r acts on V and is the transformation of the rth j step operator on $\oplus_d V$. A_j^r is zero except on the subspace V^r of V and its action on V^r is calculated from the well known fact that

$$a_j f_m = m_j^{1/2} f_{(m_1, \ldots, m_j - 1, \ldots, m_n)}$$

which gives

$$A_j^r f_m = d_j^{-1/2}(m_j - r_j)^{1/2} f_{(m - \{d_j\})}.$$

This is precisely the same as the action which Maiella defined on V^r by

$$A_j^r = d_j^{-1/2}\{(a_j a_j^*)^{-1/2} a_j\}^{d_j - r_j - 1} a_j \{(a_j a_j^*)^{-1/2} a_j\}^{r_j}.$$

As the commutation relations remain invariant under U and as A_j^r is just the transform of a_j under U, the connection between A_j^r, SU(n) and the hamiltonian follows immediately from the relations between a_j, SU(n) and the isotropic hamiltonian.

4. Conclusions

The decomposition of the anisotropic harmonic oscillator by the unitary map U shows how the degeneracy of the rationally related case is connected with representations of SU(n). However an elegant account of the degeneracy would require a group theoretical explanation or construction of U, rather than the present definition in terms of the basis.

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